Going beyond eddy viscosity: Finding a general representation of subgrid-scale stresses in large-eddy simulation

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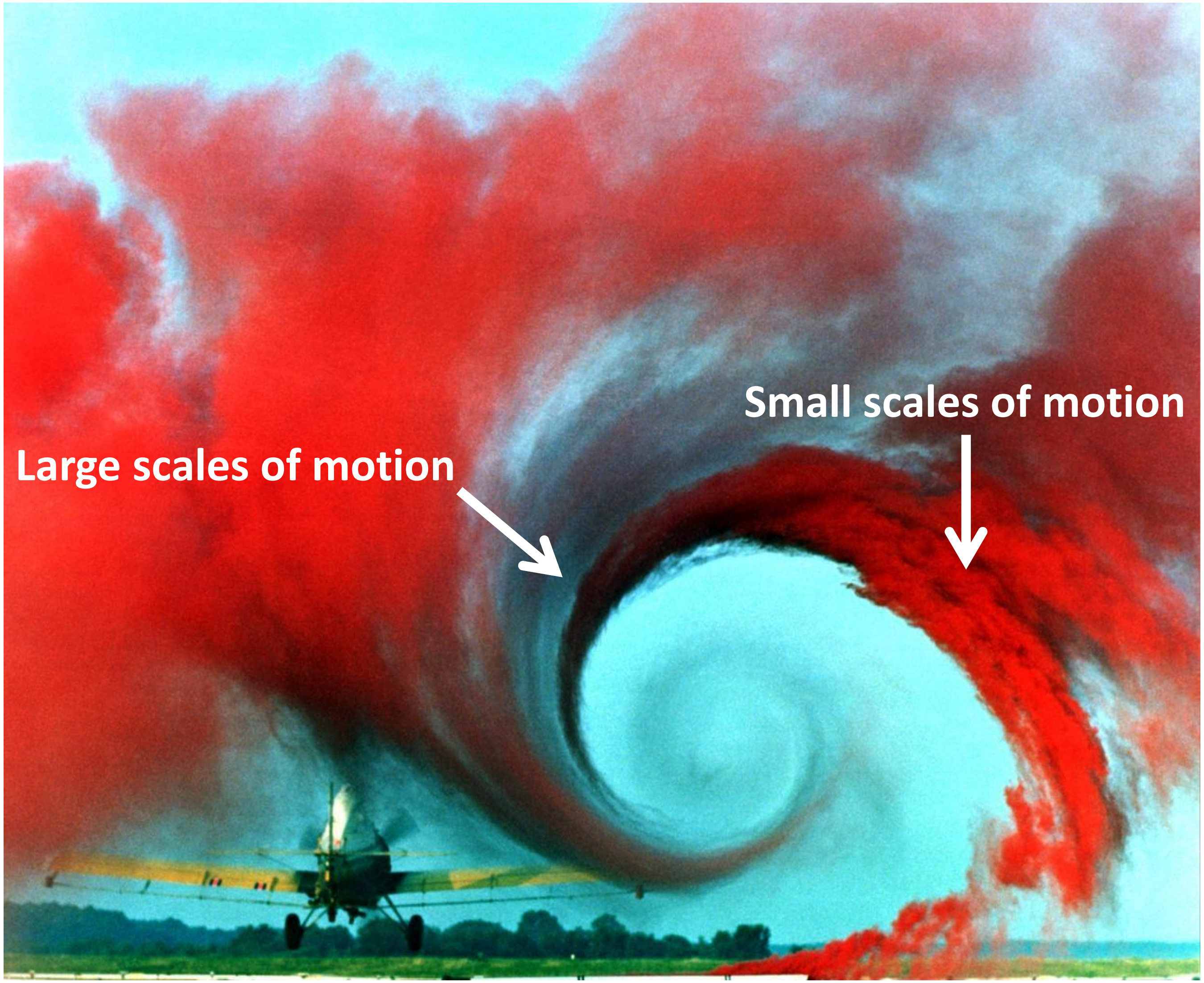
Overview

- Goal: describe and predict the behavior of turbulent flows
- Tool: large-eddy simulation

- Goal: improve on the purely dissipative description
- Tool: nonlinear model

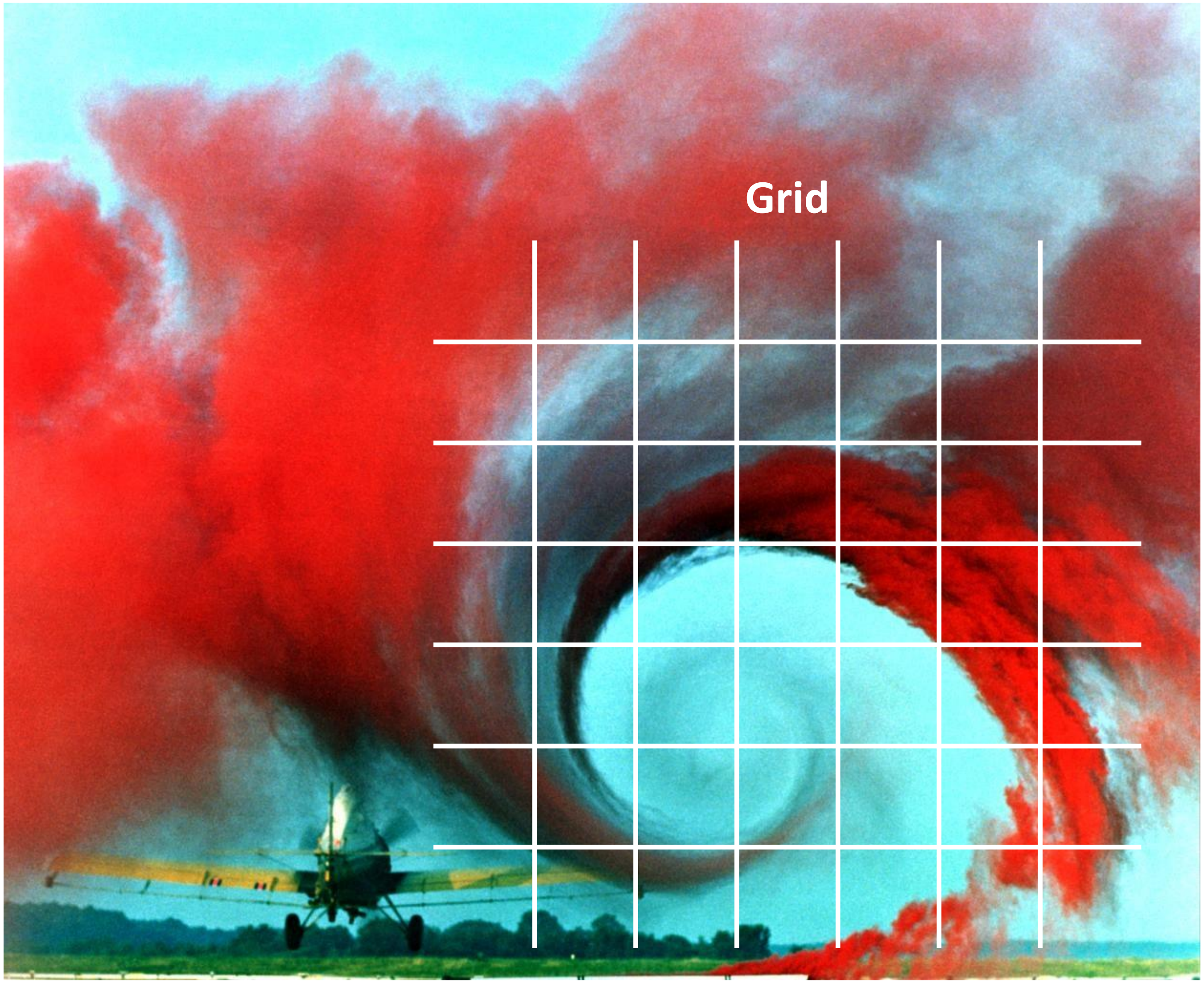
- Today's topics:
  - Construction of a general nonlinear model for large-eddy simulation
  - Discussion of physical constraints, applied to the model

Describing turbulent flows



Wake Vortex Study at Wallops Island, NASA Langley Research Center

Describing turbulent flows



Wake Vortex Study at Wallops Island, NASA Langley Research Center

Large-eddy simulation

- Filtered Navier-Stokes equations

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i \bar{u}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\tau_{ij}$$

  - Describe evolution of large-scale motions
- Closure problem
  - $\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$ unknown
  - Provide closure / subgrid-scale model:
    $\tau \approx \tau^{mod}$
    Here: in terms of $\bar{u}_i$ and derivatives

Pope (2011) *Turbulent Flows*
Sagaut (2005) *Large Eddy Simulation for Turbulent Flows*
Berselli, Iliescu and Layton (2006) *Mathematics of Large Eddy Simulation of Turbulent Flows*

Examples of subgrid-scale models

- Building blocks $\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$, $\bar{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right)$
- Eddy viscosity models

  $\tau^{mod} - \frac{1}{3}\mathrm{tr}\left(\tau^{mod}\right) I = -2\nu_e \bar{S}$

  \+ Mean dissipation of energy can be captured

  – Backscatter not captured very well, structure of τ modeled incorrectly
- The gradient / nonlinear model

  $\tau^{mod} = C\, \nabla\bar{u}\, (\nabla\bar{u})^T = C(\bar{S}^2 - \bar{\Omega}^2 - (\bar{S}\,\bar{\Omega} - \bar{\Omega}\bar{S}))$

  \+ Structure of τ captured better

  – Not dissipative enough, unstable
- More general nonlinear models

Winckelmans et al. (2001), *Phys. Fluids* **13**, 5
Vreman, Geurts and Kuerten (1996), *Theor. Comp. Fluid Dyn.* **8**, 309
Tao et al. (2002), *J. Fluid Mech.* **457**, 35
Horiuti (2001), *Direct and Large Eddy-Simulation IV*, 67

Assumptions

- Filtered velocity gradient determines subgrid stresses

  $\tau^{mod} = f(\bar{S}, \bar{\Omega})$

- Condition of isotropy

  $Q\tau^{mod}Q^T = f(Q\bar{S}Q^T, Q\bar{\Omega}Q^T)$

  – Orthogonal transformation of coordinate system: Q

- Resulting model: general polynomial
  – Terms: $\bar{S}^{m_1}\bar{\Omega}^{n_1}\bar{S}^{m_2}\bar{\Omega}^{n_2}$ ...
  – Coefficients: depending on scalar invariants of terms

Pope (1975), *J. Fluid Mech*. **72**, 331
Lund and Novikov (1992), *CTR, Ann. Res. Briefs, 27*
Gatski and Jongen (2000), *Progr. Aerosp. Sci.*, **36**, 655

Cayley-Hamilton theorem

- Resulting model: $\tau^{mod} = \sum_{i=0}^{10} \alpha_i T_i$
  - Terms T_i:

$$T_0 = I \qquad T_3 = \bar{\Omega}^2 \qquad T_7 = \bar{\Omega}\,\bar{S}\,\bar{\Omega}^2 - \bar{\Omega}^2\bar{S}\,\bar{\Omega}$$
$$T_1 = \bar{S} \qquad T_4 = \bar{S}\,\bar{\Omega} - \bar{\Omega}\,\bar{S} \qquad T_8 = \bar{S}\,\bar{\Omega}\,\bar{S}^2 - \bar{S}^2\bar{\Omega}\,\bar{S}$$
$$T_2 = \bar{S}^2 \qquad T_5 = \bar{S}^2\bar{\Omega} - \bar{\Omega}\,\bar{S}^2 \qquad T_9 = \bar{S}^2\bar{\Omega}^2 + \bar{\Omega}^2\bar{S}^2$$
$$T_6 = \bar{S}\,\bar{\Omega}^2 + \bar{\Omega}^2\bar{S} \qquad T_{10} = \bar{\Omega}\,\bar{S}^2\bar{\Omega}^2 - \bar{\Omega}^2\bar{S}^2\bar{\Omega}$$

  - Coefficients α_i depending on scalar invariants:

$$I_1 = \mathrm{tr}(\bar{S}^2) \qquad I_3 = \mathrm{tr}(\bar{S}^3) \qquad I_5 = \mathrm{tr}(\bar{S}^2\bar{\Omega}^2)$$
$$I_2 = \mathrm{tr}(\bar{\Omega}^2) \qquad I_4 = \mathrm{tr}(\bar{S}\,\bar{\Omega}^2) \qquad I_6 = \mathrm{tr}(\bar{S}^2\bar{\Omega}^2\bar{S}\,\bar{\Omega})$$

- Complexity of model
  - Practical simulations
  - Number of terms

Pope (1975), *J. Fluid Mech.* **72**, 331
Lund and Novikov (1992), *CTR, Ann. Res. Briefs, 27*
Gatski and Jongen (2000), *Progr. Aerosp. Sci.*, **36**, 655

Gram-Schmidt orthogonalization

- Process

$$T_0 = I \qquad \longrightarrow \qquad T_0' = I$$

$$T_1 = \bar{S} \qquad T_1' = \bar{S}$$

$$T_2 = \bar{S}^2 \qquad T_2' = \bar{S}^2 - \frac{1}{3} I_1 T_0' - \frac{I_3}{I_1} T_1'$$

$$T_3 = \bar{\Omega}^2 \qquad T_3' = \bar{\Omega}^2 - \frac{1}{3} I_2 T_0' - \frac{I_4}{I_2} T_1' - \cdots T_2'$$

$$\vdots \qquad \vdots$$

Beyond eddy viscosity

- Results
  - T_8', T_9', T_{10}' always zero
  - At least two other tensors also zero

  Usually a general representation of τ^{mod}

  - Independence of terms, mathematically and physically

Results

- General nonlinear model: $\tau^{mod} = \sum_{i=0}^{7} \alpha_i' T_i'$
  - At least two tensors zero: usually a general representation of τ^{mod}
  - Independence of terms, mathematically and physically

- Goal:
  - Understand the functional dependence of the coefficients on flow properties
  - Understand the behavior of the different terms

Symmetry requirements

- Symmetries of the Navier-Stokes equations
  - Rotations and reflections
  - Scaling transformations
  - Etc.
- Assumptions
  - Symmetries apply to *filtered* Navier-Stokes equations
- Goal
  - Make model invariant
- Importance
  - Be able to represent solutions well

Razafindralandy, Hamdouni and Oberlack (2007), *Eur. J. Mech. B-Fluid.*, **26**, 531

Symmetry requirements

- Condition of isotropy

  $$Q\tau^{mod}Q^T = f(Q\bar{S}Q^T, Q\bar{\Omega}Q^T)$$

  - Orthogonal transformation of the coordinate frame: Q
- Importance
  - Independence of observation angle
- Model constraint
  - Coefficients depend on scalar quantities:

$$\alpha_i' = \alpha_i'(I_1, I_2, \dots, I_6)$$

$$I_1 = \mathrm{tr}(\bar{S}^2)$$

$$\dots$$

Symmetry requirements

- Scaling symmetry

$\hat{t} = e^{2a} t \qquad \hat{\bar{u}}_i = e^{-a+b} \hat{\bar{u}}_i \qquad \hat{\nu} = e^{2b} \nu$

$\hat{x}_i = e^{a+b} x_i \qquad \hat{\bar{p}} = e^{-2a+2b} \bar{p}$

- Condition

$\hat{\tau}^{mod} = e^{-2a+2b} \tau^{mod}$

- Importance
  - Represent self-similar solutions well
- Model constraint
  - Units of the model coefficients, α_i'

Symmetry requirements

- Scaling symmetry constraint (I)

$$\underbrace{\tau^{mod}}_{m^2 s^{-2}} = \underbrace{\nu_0}_{m^2 s^{-1}} \underbrace{f_0(I_1, I_2, \dots, I_6)}_{s^{-1}} \underbrace{I}_{1}$$

$$+ \underbrace{\nu_1}_{m^2 s^{-1}} \underbrace{f_1(I_1, I_2, \dots, I_6)}_{1} \underbrace{\bar{S}}_{s^{-1}}$$

$$+ \underbrace{\nu_2}_{m^2 s^{-1}} \underbrace{f_2(I_1, I_2, \dots, I_6)}_{s} \underbrace{(\bar{S}^2 - \cdots)}_{s^{-2}} + \cdots$$

$I_1 = \mathrm{tr}(\bar{S}^2)$

…

- Dependence of α_i' on $I_1, I_2, \dots, I_6$ partly determined
  - Units of f_i known
  - Limiting cases

Symmetry requirements

- Scaling symmetry constraint (II)

$$\underbrace{\tau^{mod}}_{m^2 s^{-2}} = \underbrace{c_0}_{1}\,\underbrace{\delta^2}_{m^2}\,\underbrace{f_0(I_1, I_2, \dots, I_6)}_{s^{-2}}\,\underbrace{I}_{1}$$

$$+\underbrace{c_1}_{1}\,\underbrace{\delta^2}_{m^2}\,\underbrace{f_1(I_1, I_2, \dots, I_6)}_{s^{-1}}\,\underbrace{\bar{S}}_{s^{-1}}$$

$$+\underbrace{c_2}_{1}\,\underbrace{\delta^2}_{m^2}\,\underbrace{f_2(I_1, I_2, \dots, I_6)}_{1}\,\underbrace{(\bar{S}^2 - \cdots)}_{s^{-2}} + \cdots$$

$I_1 = \mathrm{tr}(\bar{S}^2)$

...

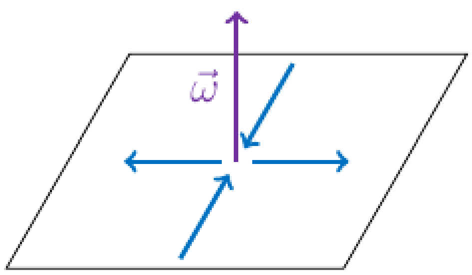
- Dependence of α_i' on I_1, I_2, ..., I_6 partly determined
  - Units of f_i known
  - Limiting cases
- Requires dynamic procedure due to external length scale δ

The vanishing subgrid dissipation requirement

- Vreman's requirement for eddy viscosity models

  $\mathrm{tr}(\tau\bar{S}) = \tau : \bar{S} = 0 \;\rightarrow\; \mathrm{tr}\left(\tau^{mod}\bar{S}\right) = 0$

- Procedure
  - Choose a certain flow / velocity gradient $\left[\frac{\partial u_i}{\partial x_j}\right]$, e.g. $\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$
  - Compute τ and $\bar{S}$
  - Compute subgrid dissipation $\mathrm{tr}(\tau\bar{S})$
  - If zero, require $\mathrm{tr}\left(\tau^{mod}\bar{S}\right) = 0$
- Results
  - Determines zeros of $\alpha_1'(I_1, I_2, \ldots, I_6) = \mathrm{tr}\left(\tau^{mod}\bar{S}\right)/I_1$



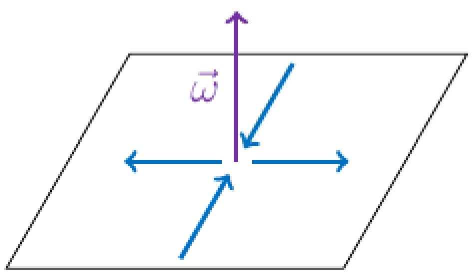
Vreman, A.W. (2004), *Phys. Fluids* **16**, 3670

The vanishing subgrid force requirement

- Extending Vreman's requirement

$$-\frac{\partial}{\partial x_j}\tau_{ij} = 0 \rightarrow -\frac{\partial}{\partial x_j}\tau_{ij}^{mod} = 0$$

- Procedure
  - Choose a certain flow / velocity gradient $\left[\frac{\partial u_i}{\partial x_j}\right]$, e.g. $\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$
  - Compute subgrid force $\frac{\partial}{\partial x_j}\tau_{ij}$
  - If zero, require $\frac{\partial}{\partial x_j}\tau_{ij}^{mod} = 0$
- Results
  - Determines zeros of $\alpha_1'(I_1, I_2, \ldots, I_6)$
  - Provides relation between α_0', α_2', α_4' for certain flows



Class of nonlinear models

- We have found a class of subgrid-scale models
  - nonlinear in $\bar{S}$ and $\bar{\Omega}$
  - consisting of a finite number of independent terms
  - usually providing a representation of τ
  - satisfying several physical constraints

$$\tau^{mod} = \sum_{i=0}^{7} \alpha_i' T_i'$$

- We have discussed several physical model constraints
  - satisfying the symmetries of the Navier-Stokes equations: $\alpha_i'(I_1, I_2, \dots, I_6)$, with proper units
  - vanishing subgrid dissipation and subgrid force for simple flows: zeros of α_1', relation between α_0', α_2', α_4'

Analysis of existing models

- Vreman's model

$$\tau^{mod} - \tfrac{1}{3}\mathrm{tr}\left(\tau^{mod}\right) I = \alpha_1' \bar{S} \qquad \alpha_1' = -2c\delta^2 \sqrt{\frac{(I_1-I_2)^2-\frac{1}{2}(I_1+I_2)^2+8I_5}{I_1-I_2}}$$

  - can only satisfy scale invariance if c is determined dynamically
  - α_1' vanishes when $\mathrm{tr}(\tau\bar{S}) = 0$ ✓

- QR model

$$\tau^{mod} - \tfrac{1}{3}\mathrm{tr}\left(\tau^{mod}\right) I = \alpha_1' \bar{S} \qquad \alpha_1' = -2C_\delta \frac{\max(0, I_3)}{I_1}$$

  - can only satisfy scale invariance if C_δ is determined dynamically
  - α_1' vanishes for all two-dimensional strains ✗

Vreman, A.W. (2004), *Phys. Fluids* **16**, 3670
Verstappen, Rozema, Bae (2014), *CTR Proc. Summ. Progr.*, 417

Outlook

- Expectations of our class of subgrid-scale models
  - Better representation of subgrid-scale stresses than eddy viscosity models
  - Capture other than dissipative effects

- Future research
  - Discover which processes are represented by the different nonlinear model terms
  - Run flow simulations with different choices for the model coefficients satisfying the discussed constraints

- Any suggestions?
  - Additional physical constraints?
  - Interpretation of the nonlinear model terms?

Thank you for your attention!

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