The higher-order gradient model for large-eddy simulation of turbulent flows

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Overview

- Research aim
- Large-eddy simulation
- The gradient model
- The higher-order gradient model
- Numerical results
- Conclusions and future work

Description of incompressible flows

- Navier-Stokes equations

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

- Filtered Navier-Stokes equations

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\underbrace{(\overline{u_i u_j} - \bar{u}_i \bar{u}_j)}_{\tau_{ij}}$$

Sagaut, P. (2005) *Large Eddy Simulation for Turbulent Flows*

Description of incompressible flows

- Filtering
  - Convolution product:

$$\overline{u}_i(\vec{x},t) = \int u_i(\vec{x}',t)G(\vec{x}-\vec{x}')\,\mathrm{d}\vec{x}'$$

  - Box filter

$$G(\vec{x}-\vec{x}') = \frac{1}{\delta^3}\begin{cases} 1 & \text{if } |\vec{x}-\vec{x}'| \leq \dfrac{\delta}{2} \\ 0 & \text{otherwise} \end{cases}$$

  - Filter length: δ

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The gradient model

- "Approximate deconvolution method based on wavenumber asymptotics"
- Derivation

$$\tau_{ij} = \underbrace{\overline{u_i u_j} - \bar{u}_i \bar{u}_j}_{\#1} \qquad \qquad \tau_{ij} = \underbrace{\overline{\bar{u}_i \bar{u}_j} + \overline{\bar{u}_i u_j'} + \overline{u_i' \bar{u}_j} + \overline{u_i' u_j'} - \bar{u}_i \bar{u}_j}_{\#2}$$

$\downarrow$ $\mathcal{F}$ Fourier transform $\downarrow$

$\downarrow$ $\sum_{m=0}^{n} \ldots \delta^n$ Taylor expansion in the filter length $\downarrow$

$\downarrow$ $\mathcal{F}^{-1}$ Inverse Fourier transform $\downarrow$

$$\underbrace{\tau_{ij} = f(u_k) + \mathcal{O}(\delta^{n+2})}_{\#1} \qquad \qquad \underbrace{\tau_{ij} = g(\bar{u}_k) + \mathcal{O}(\delta^{n+2})}_{\#2}$$

The gradient model

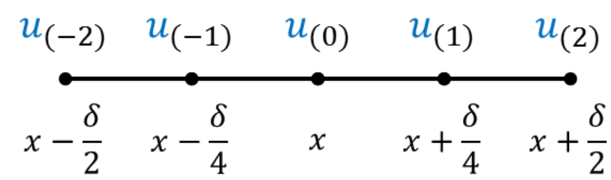
$$\underbrace{\tau_{ij} = \frac{1}{12}\delta^2 \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k} + \mathcal{O}(\delta^4)}_{\#1} \qquad \underbrace{\tau_{ij} = \frac{1}{12}\delta^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} + \mathcal{O}(\delta^4)}_{\#2}$$

- Problems
  - Numerically unstable
  - Turbulent stresses $\overline{u_i' u_j'} \sim \mathcal{O}(\delta^4)$, hence **not** modeled
- Solutions
  - Clark model / tensor-diffusivity model with eddy viscosity
  - Higher-order gradient model
- Current work
  - Approach #1, as a pilot for #2

Berselli, L.C. and Iliescu, T. , J. Comput. Appl. Math. **159**, 411-430 (2003)
Winckelmans, G.S. et al., Phys. Fluids **13**, 5 (2001)
Vreman, B., Geurts, B. and Kuerten, H., Theor. Comp. Fluid Dyn. 8, 309-342 (1996)

Numerical setup

- Discretization of model
  - Boole's numerical integration rule (1D)



$$\bar{u} = \frac{1}{\delta}\int_{x'=x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} u(x')\,\mathrm{d}x'$$

$$= \tfrac{7}{90}u\left(x-\tfrac{\delta}{2}\right) + \tfrac{32}{90}u\left(x-\tfrac{\delta}{4}\right) + \tfrac{12}{90}u(x) + \tfrac{32}{90}u\left(x+\tfrac{\delta}{4}\right) + \tfrac{7}{90}u\left(x+\tfrac{\delta}{2}\right) + \mathcal{O}(\delta^6)$$

$$= \sum_{l=-2}^{2} \alpha_{(l)} u_{(l)} + \mathcal{O}(\delta^6)$$

  - Subfilter-scale model

$$\tau_{11} = \overline{uu} - \bar{u}\bar{u} = \sum_{l=-2}^{2} \alpha_{(l)} u_{(l)}^2 - \left[\sum_{l=-2}^{2} \alpha_{(l)} u_{(l)}\right]^2 + \mathcal{O}(\delta^6)$$

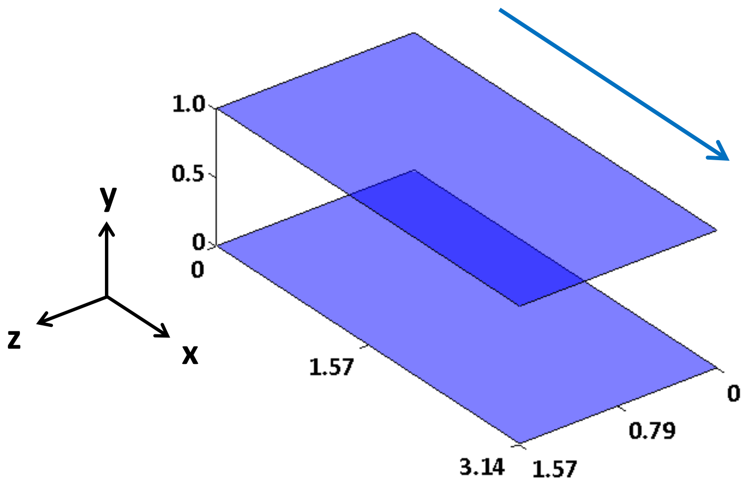
  - Extension to 3D: sum in each direction, using 5^3 points
  - Choice of filter length: $\delta = 4h$

Numerical setup

- Discretization
  - Finite-volume method on a staggered grid
  - Explicit second-order accurate time-stepping method
  - Fourth-order symmetry-preserving spatial discretization
  - Higher-order gradient model discretization

Numerical test case

- Plane-channel flow
  - Periodic in x and z-directions
  - Mass flow prescribed
  - Friction Reynolds number $Re_\tau = 590$

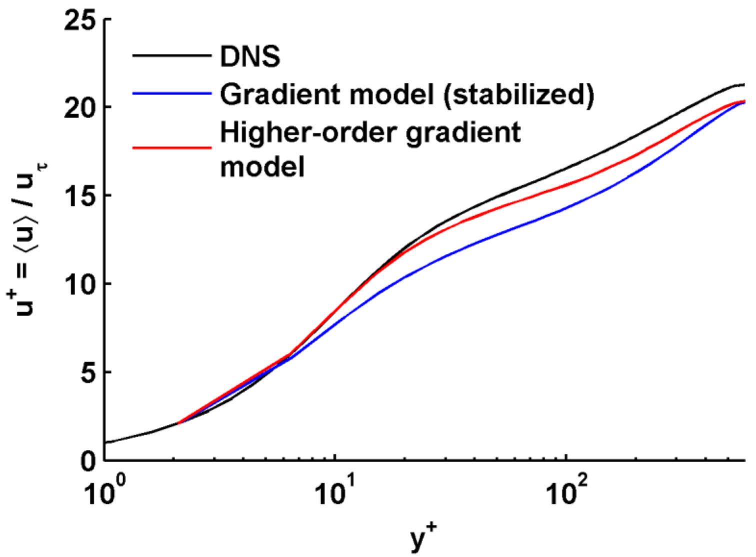


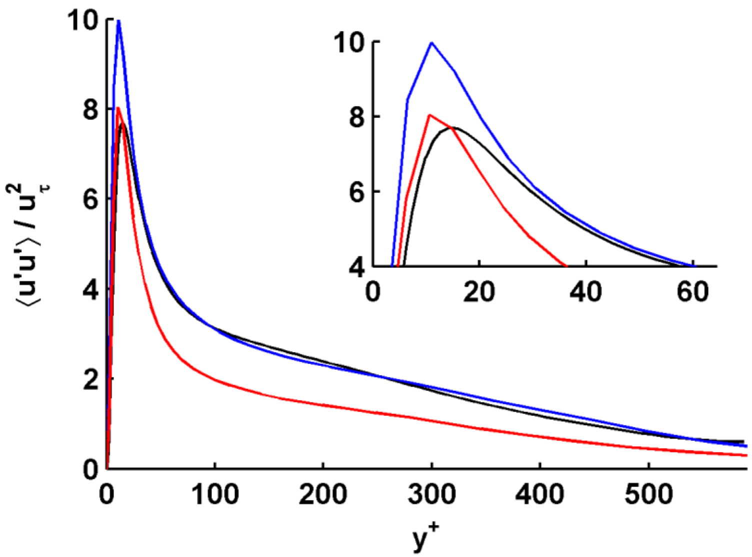
- Grid
  - LES: 64^3 nonuniform grid for a $\pi * 1 * \pi/2$ channel compared to
  - DNS: $384 * 257 * 384$ grid

Moser, R.D., Kim, J. and Mansour, N.N. , J. Phys. Fluids **11**, 943-945 (1999)

Numerical results

- Numerical results
  - Velocity profile: u^+
  - Reynolds stress profile: $\langle u'u' \rangle$





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Conclusions & Future work

- The higher-order gradient model
  - approximate deconvolution method
  - no model constant
  - error smaller than the turbulent stresses
  - tested in a pilot study (approach #1)
  - numerically stable (so far)

- Future work
  - perform stability analysis of model equations
  - reduce the number of points (5^3) used in the discretization
  - use the ‘true’ deconvolution model $\tau_{ij}(\overline{u}_k)$ (approach #2)

Thank you for your attention!

Model equations

- Filtered Navier-Stokes equations

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\underbrace{(\overline{u_i u_j} - \bar{u}_i \bar{u}_j)}_{\tau_{ij}}$$

- Model equations (approach #1)
  - Drop bars

$$\frac{\partial v_i}{\partial x_i} = 0$$

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j}(v_i v_j) = -\frac{\partial q}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 v_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\underbrace{\tau_{ij}(v_k)}_{\#1}$$

The higher-order gradient model

- Approach #1 with box filter

$$\tau_{ij} = \frac{1}{12}\delta^2\left[\frac{\partial u_i}{\partial x_1}\frac{\partial u_j}{\partial x_1} + \cdots\right]$$

$$+\frac{1}{1440}\delta^4\left[\; 2\frac{\partial^2 u_i}{\partial x_1^2}\frac{\partial^2 u_j}{\partial x_1^2} + \cdots \right.$$ Pure second derivatives

$$10\frac{\partial^2 u_i}{\partial x_1 \partial x_2}\frac{\partial^2 u_j}{\partial x_1 \partial x_2} + \cdots$$ Mixed second derivatives

$$3\frac{\partial u_i}{\partial x_1}\frac{\partial^3 u_j}{\partial x_1^3} + \cdots$$ Pure third derivatives

$$\left. 5\frac{\partial u_i}{\partial x_1}\frac{\partial^3 u_j}{\partial x_1 \partial x_2^2} + \cdots \;\right]$$ Mixed third derivatives

$$+\mathcal{O}(\delta^6)$$

The higher-order gradient model

- Approach #2 with box filter

$$
\begin{aligned}
\tau_{ij} = \frac{1}{12}\delta^2 &\left[\frac{\partial \bar{u}_i}{\partial x_1}\frac{\partial \bar{u}_j}{\partial x_1} + \cdots\right] \\
+ \frac{1}{1440}\delta^4 &\left[\; 2\frac{\partial^2 \bar{u}_i}{\partial x_1^2}\frac{\partial^2 \bar{u}_j}{\partial x_1^2} + \cdots \right. && \text{Pure second derivatives} \\
& \quad 10\frac{\partial^2 \bar{u}_i}{\partial x_1 \partial x_2}\frac{\partial^2 \bar{u}_j}{\partial x_1 \partial x_2} + \cdots && \text{Mixed second derivatives} \\
& \quad \left. -2\frac{\partial u_i}{\partial x_1}\frac{\partial^3 u_j}{\partial x_1^3} + \cdots \;\right] && \text{Pure third derivatives} \\
+\mathcal{O}(\delta^6) &
\end{aligned}
$$