The higher-order gradient model for large-eddy simulation of turbulent flows

Maurits Silvis

Supervisors
Roel Verstappen
Arthur Veldman

IPAM Mathematics of Turbulence

September 18, 2014

university of groningen / faculty of mathematics and natural sciences / johann bernoulli institute for mathematics and computer science

Overview

- Research aim
- Large-eddy simulation
- The gradient model
- The higher-order gradient model
- Preliminary numerical results
- Review and future work

Description of incompressible flows

- Navier-Stokes equations

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

  - Infeasible to solve for all relevant scales (DNS) for most flows
  - Leave out information about the smallest scales (LES)

Large-eddy simulation

- Filtering

$$\overline{u}_i(\vec{x},t) = \int\int u_i(\vec{x}',t')G(\vec{x},\vec{x}',t,t')\,\mathrm{d}\vec{x}'\,\mathrm{d}t \qquad u_i = \overline{u}_i + u_i'$$

- Homogeneous spatial filter

$$\overline{u}_i(\vec{x},t) = \int u_i(\vec{x}',t)G(\vec{x}-\vec{x}')\,\mathrm{d}\vec{x}' = G * u_i(\vec{x},t)$$

- Examples of filters
  - Box filter (our choice)

$$G(\vec{x}-\vec{x}') = \frac{1}{\delta^3}\begin{cases}1 & \text{if } |\vec{x}-\vec{x}'| \leq \dfrac{\delta}{2} \\ 0 & \text{otherwise}\end{cases}$$

Filter length: δ

  - Gaussian filter
  - Spectral cutoff filter
  - More…

Sagaut (2005) *Large Eddy Simulation for Turbulent Flows*
Berselli, Iliescu and Layton (2006) *Mathematics of Large Eddy Simulation of Turbulent Flows*

Large-eddy simulation

- Filtered Navier-Stokes equations

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{u_i u_j}\right) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}$$

  – Closure problem

- Rewrite and provide closure model

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i \bar{u}_j\right) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\underbrace{\left(\overline{u_i u_j} - \bar{u}_i \bar{u}_j\right)}_{\tau_{ij}}$$

Sagaut (2005) *Large Eddy Simulation for Turbulent Flows*
Berselli, Iliescu and Layton (2006) *Mathematics of Large Eddy Simulation of Turbulent Flows*

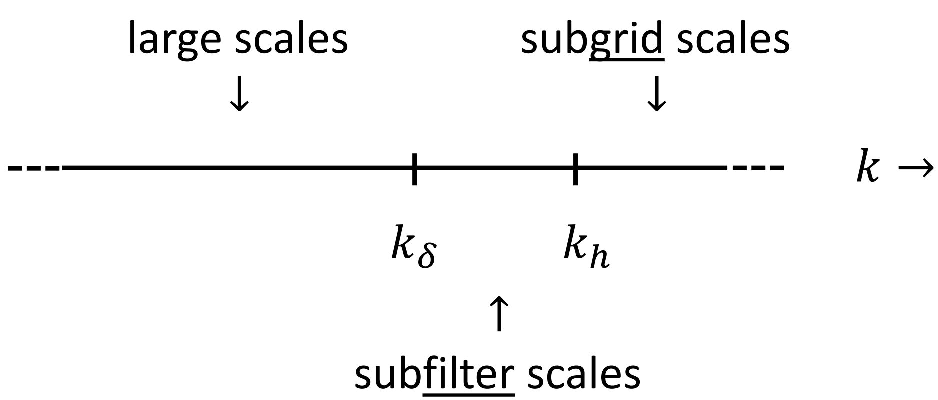
Large-eddy simulation

- Examples of closure models
  - Boussinesq hypothesis
    $\tau_{ij} - \frac{1}{3}\tau_{ii}\delta_{ij} \approx -2\nu_t \bar{S}_{ij}$
  - Bardina's scale-similarity model
    $\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j \approx \overline{\bar{u}_i \bar{u}_j} - \bar{\bar{u}}_i \bar{\bar{u}}_j$
  - Approximate deconvolution model
    $\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j \approx \overline{u_i^* u_j^*} - \bar{u}_i \bar{u}_j$ where $u_i^* \approx G^{-1} * \bar{u}_i$
  - Many more…
- Our focus: 'expansion deconvolution methods'

Sagaut (2005) *Large Eddy Simulation for Turbulent Flows*
Berselli, Iliescu and Layton (2006) *Mathematics of Large Eddy Simulation of Turbulent Flows*
Stolz and Adams (1999) Phys. Fluids **11**, 1699
Stolz, Adams and Kleiser (2001), Phys. Fluids **13**, 997
Katopodes Chow, Street and Ferziger (2000), Stanford University Technical Report 2000-K1

Large-eddy simulation

- Explicit filter
  - Box filter, Gaussian filter, …
  - Denoted by $\bar{u}_i$
- Implicit filter
  - Numerics, grid
  - Could be denoted by $\tilde{u}_i$
- Distinguishing scales



- Subfilter-scale stresses
  - May be recovered mathematically (our focus)
- Subgrid-scale stresses
  - Irreversibly lost
  - Must be modeled (not our focus)

Carati et al. (2001), J. Fluid Me ch. **441**, 119
Winckelmans et al. (2001), Phys. Fluids **13**, 5

The expansion deconvolution method

- Derivation

$$\tau_{ij} = \underbrace{\overline{u_i u_j} - \bar{u}_i \bar{u}_j}_{\#1} \qquad \tau_{ij} = \underbrace{\overline{\bar{u}_i \bar{u}_j} + \overline{\bar{u}_i u_j'} + \overline{u_i' \bar{u}_j} + \overline{u_i' u_j'} - \bar{u}_i \bar{u}_j}_{\#2}$$

↓	$\mathcal{F}$	Fourier transform
↓	$\sum_{m=0}^{n} \ldots \delta^m$	Truncated Taylor expansion in the filter length
↓	$\mathcal{F}^{-1}$	Inverse Fourier transform

$$\underbrace{\tau_{ij} = f(u_k) + \mathcal{O}(\delta^{n+2})}_{\#1} \qquad \underbrace{\tau_{ij} = g(\bar{u}_k) + \mathcal{O}(\delta^{n+2})}_{\#2}$$

  - Consistent approximation of actual sub<u>filter</u>-scale stresses
  - An example: the gradient model

The gradient model

- The gradient model for the subfilter stress

$$\underbrace{\tau_{ij} = \frac{1}{12}\delta^2 \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k} + \mathcal{O}(\delta^4)}_{\#1} \qquad \underbrace{\tau_{ij} = \frac{1}{12}\delta^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} + \mathcal{O}(\delta^4)}_{\#2}$$

- Promises
  - Good correlations with actual subfilter-scale stresses
- Problems (1)
  - Linearly unstable when supplemented to Burgers' equation
- Possible remedies (1)
  - Add (minimal amount of) eddy viscosity
  - Regularize the model by filtering it

Winckelmans et al. (2001), Phys. Fluids **13**, 5
Katopodes Chow, Street and Ferziger (2000), Stanford University Technical Report 2000-K1
Vreman, Geurts and Kuerten (1996), Theor. Comp. Fluid Dyn. 8, 309-342
Berselli, Iliescu and Layton (2006) *Mathematics of Large Eddy Simulation of Turbulent Flows*
Berselli and Iliescu (2003), J. Comput. Appl. Math. **159**, 411-430

The gradient model

- The gradient model for the subfilter stress

$$\underbrace{\tau_{ij} = \frac{1}{12}\delta^2 \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k} + \mathcal{O}(\delta^4)}_{\#1} \qquad \underbrace{\tau_{ij} = \frac{1}{12}\delta^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} + \mathcal{O}(\delta^4)}_{\#2}$$

- Promises
  - Good correlations with actual subfilter-scale stresses
- Problems (2)
  - Turbulent stresses $\overline{u_i' u_j'} \sim \mathcal{O}(\delta^4)$, hence **not** modeled
- Possible remedies (2)
  - Add higher-order contribution from expansion method

The higher-order gradient model

- The higher-order gradient model for the subfilter stress

$$\underbrace{\tau_{ij} = \frac{1}{12}\delta^2 \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k} + f(u_k)\delta^4 + \mathcal{O}(\delta^6)}_{\#1} \qquad \underbrace{\tau_{ij} = \frac{1}{12}\delta^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} + g(\bar{u}_k)\delta^4 + \mathcal{O}(\delta^6)}_{\#2}$$

- Promises
  - Better correlation with actual subfilter-scale stresses
- Problems
  - 'More linearly unstable' when supplemented to Burgers' equation: no finite amount of eddy viscosity can damp perturbations.
- Possible remedies
  - Adapt/regularize the model
- Preliminary numerical tests
  - Approach #1. Seems numerically stable?

Katopodes Chow, Street and Ferziger (2000), Stanford University Technical Report 2000-K1
Vreman, Geurts and Kuerten (1996), Theor. Comp. Fluid Dyn. 8, 309-342

Numerical setup

- Filtered Navier-Stokes equations

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\underbrace{(\overline{u_i u_j} - \bar{u}_i \bar{u}_j)}_{\tau_{ij}}$$

- Model equations (approach #1)

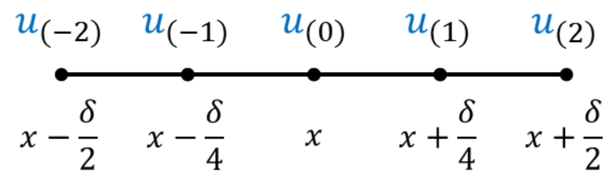
$$\frac{\partial \bar{v}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{v}_i \bar{v}_j) = -\frac{\partial \bar{q}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\underbrace{\left(\frac{1}{12}\delta^2 \frac{\partial v_i}{\partial x_k}\frac{\partial v_j}{\partial x_k} + f(v_k)\delta^4\right)}_{\#1}$$

  - Drop bars
  - Assume $v \approx \bar{u}$, $q \approx \bar{p}$

Numerical setup

- Discretization of model (approach #1)
  - Boole's numerical integration rule (1D)



$$\bar{u} = \frac{1}{\delta}\int_{x'=x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} u(x')\,\mathrm{d}x'$$

$$= \tfrac{7}{90}u\left(x-\tfrac{\delta}{2}\right) + \tfrac{32}{90}u\left(x-\tfrac{\delta}{4}\right) + \tfrac{12}{90}u(x) + \tfrac{32}{90}u\left(x+\tfrac{\delta}{4}\right) + \tfrac{7}{90}u\left(x+\tfrac{\delta}{2}\right) + \mathcal{O}(\delta^6)$$

$$= \sum_{l=-2}^{2} \alpha_{(l)} u_{(l)} + \mathcal{O}(\delta^6)$$

  - Subfilter-scale model

$$\tau_{11} = \overline{uu} - \bar{u}\bar{u} = \sum_{l=-2}^{2} \alpha_{(l)} u_{(l)}^2 - \left[\sum_{l=-2}^{2} \alpha_{(l)} u_{(l)}\right]^2 + \mathcal{O}(\delta^6)$$

  - Extension to 3D: sum in each direction, using 5^3 points
  - Choice of filter length: $\delta = 4h$

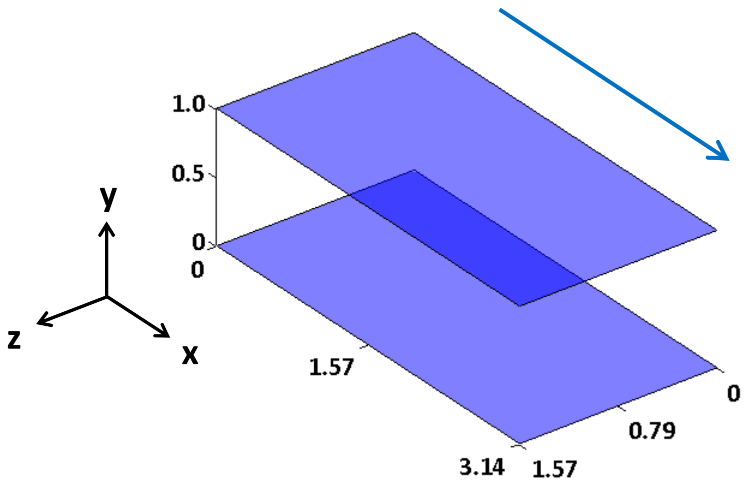
Numerical setup

- Discretization recipe
  - Finite-volume method on a staggered grid
  - Explicit second-order accurate time-stepping method
  - Fourth-order symmetry-preserving spatial discretization
  - Higher-order gradient model discretization

Verstappen and Veldman (2003), J. Comp. Phys. **187**, 343

Numerical test case

- Turbulent plane channel flow
  - Periodic in x and z-directions
  - Mass flow prescribed
  - Friction Reynolds number $Re_\tau = 590$

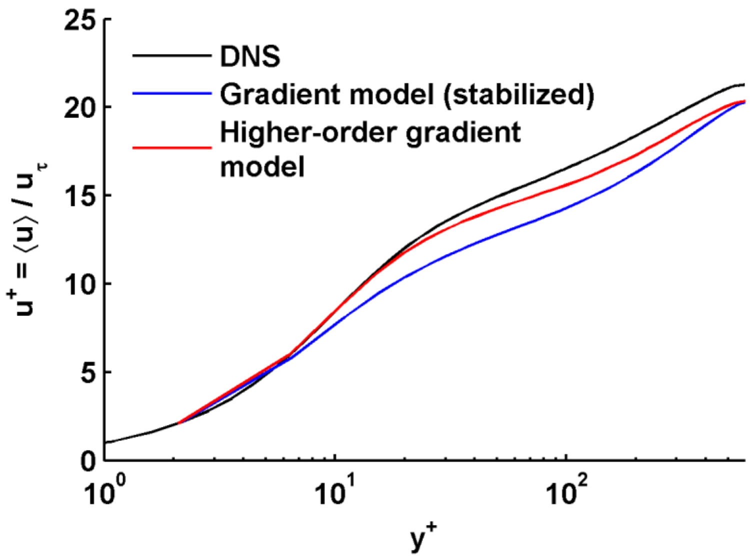


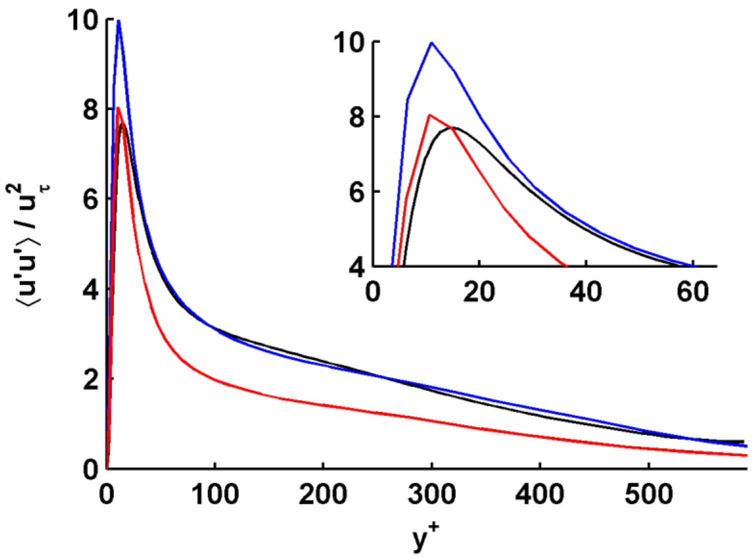
- Grid
  - LES: 64^3 nonuniform grid for a $\pi * 1 * \pi/2$ channel compared to
  - DNS: $384 * 257 * 384$ grid

Moser, Kim and Mansour (1999), J. Phys. Fluids **11**, 943-945

Preliminary numerical results

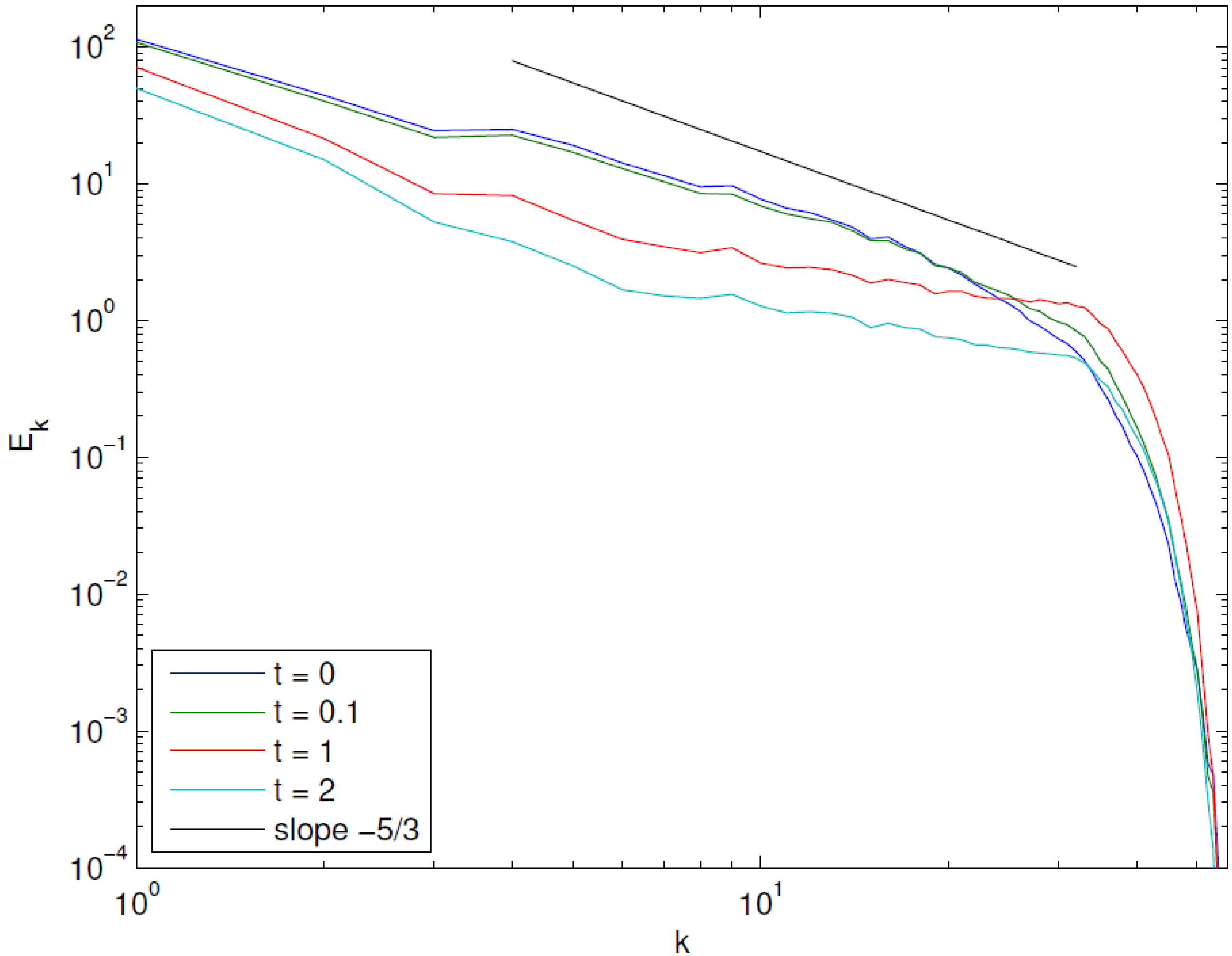
- Turbulent plane channel flow
  - Velocity profile: u^+
  - Reynolds stress profile: $\langle u'u' \rangle$





Preliminary numerical results

- Homogeneous isotropic turbulence



?

Courtesy of Wybe Rozema

Review

- Large-eddy simulation
  - Filtering
  - Closure problem
  - Closure models
- Expansion deconvolution method
  - Mathematically recover subfilter-scale stresses
  - Gradient model
  - Higher-order gradient model
- Preliminary numerical results
  - Velocity and Reynolds stress profiles for a turbulent channel flow
  - Energy spectrum for homogeneous isotropic turbulence

Future work and questions

- Future work
  - Use the ‘true’ deconvolution model $\tau_{ij}(\overline{u}_k)$ (approach #2)
  - Adapt expansion deconvolution-type models to stabilize them
  - And..
- Questions
  - (Global) dissipation properties reproduced well by most models, how to properly model dispersive effects?
    - Shape and scaling laws of spectra?
    - Correlation between subfilter/subgrid stress tensors from models and from DNS?
    - …

Thank you for your attention!

The higher-order gradient model

- Approach #1 with box filter

$$\tau_{ij} = \frac{1}{12}\delta^2\left[\frac{\partial u_i}{\partial x_1}\frac{\partial u_j}{\partial x_1} + \cdots\right]$$

$$+\frac{1}{1440}\delta^4\left[\; 2\frac{\partial^2 u_i}{\partial x_1^2}\frac{\partial^2 u_j}{\partial x_1^2} + \cdots \right.$$ Pure second derivatives

$$10\frac{\partial^2 u_i}{\partial x_1 \partial x_2}\frac{\partial^2 u_j}{\partial x_1 \partial x_2} + \cdots$$ Mixed second derivatives

$$3\frac{\partial u_i}{\partial x_1}\frac{\partial^3 u_j}{\partial x_1^3} + \cdots$$ Pure third derivatives

$$\left. 5\frac{\partial u_i}{\partial x_1}\frac{\partial^3 u_j}{\partial x_1 \partial x_2^2} + \cdots \;\right]$$ Mixed third derivatives

$$+\mathcal{O}(\delta^6)$$

The higher-order gradient model

- Approach #2 with box filter

$$
\begin{aligned}
\tau_{ij} &= \frac{1}{12}\delta^2\left[\frac{\partial \bar{u}_i}{\partial x_1}\frac{\partial \bar{u}_j}{\partial x_1} + \cdots\right] \\
&+ \frac{1}{1440}\delta^4\Bigg[\; 2\frac{\partial^2 \bar{u}_i}{\partial x_1^2}\frac{\partial^2 \bar{u}_j}{\partial x_1^2} + \cdots \\
&\qquad 10\frac{\partial^2 \bar{u}_i}{\partial x_1 \partial x_2}\frac{\partial^2 \bar{u}_j}{\partial x_1 \partial x_2} + \cdots \\
&\qquad -2\frac{\partial u_i}{\partial x_1}\frac{\partial^3 u_j}{\partial x_1^3} + \cdots \;\Bigg] \\
&+ \mathcal{O}(\delta^6)
\end{aligned}
$$

Pure second derivatives

Mixed second derivatives

Pure third derivatives