# How to model subgrid-scale effects on large-scale energy distribution in large-eddy simulation of turbulent flows?

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**Introduction** We study a general subgrid-scale model that is nonlinear in the velocity gradient. As this model forms an extension of the commonly used eddy-viscosity and gradient models, we wonder whether it captures the transfer of energy from large to small scales and vice versa, as well as the effects of small-scale motions on the distribution of energy among large scales. Further, we ask how the model can be adjusted to provide an improved representation of subgrid-scale stresses.

## Large-eddy simulation

### Filtered Navier-Stokes equations

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}\tau_{ij}$$

- Subgrid-scale stress tensor: $\tau_{ij} = \overline{u_i u_j} - \bar{u}_i\bar{u}_j$
- Closure problem: model $\tau^{mod} \approx \tau$

### Model building blocks

Rate of strain: $\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$

Rate of rotation: $\bar{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right)$

## Examples of subgrid-scale models

### Eddy-viscosity models

$$\tau^{mod} - \tfrac{1}{3}\mathrm{tr}\left(\tau^{mod}\right)I = -2\nu_e\bar{S}$$

Energy transfer from large to small scales:

$$\mathcal{P}^{mod} = -\mathrm{tr}\left(\tau^{mod}\bar{S}\right) = 2\nu_e\mathrm{tr}(\bar{S}^2)$$

+ Mean dissipation captured for $\nu_e > 0$
– Backward scatter only possible for $\nu_e < 0$
– Stress structure imposed incorrectly

### The gradient model

$$\tau^{mod} = C(\bar{S}^2 - \bar{\Omega}^2 - (\bar{S}\,\bar{\Omega} - \bar{\Omega}\,\bar{S}))$$

Energy transfer from large to small scales:

$$\mathcal{P}^{mod} = -\mathrm{tr}\left(\tau^{mod}\bar{S}\right) = -C\left(\mathrm{tr}(\bar{S}^3) - \mathrm{tr}(\bar{S}\,\bar{\Omega}^2)\right)$$

+ Represents forward and backward scatter
+ Stress structure resembled better
– Not enough forward scatter, unstable

### Mixed models

Linear combination of the above models

+ Stable model that represents forward and backward scatter
– Somewhat *ad hoc* combination of models

## General nonlinear subgrid-scale model

### Assumption

$$\tau^{mod} - \tfrac{1}{3}\mathrm{tr}\left(\tau^{mod}\right)I = f(\bar{S},\bar{\Omega})$$

### Cayley-Hamilton theorem

$$\begin{aligned}\tau^{mod} &= c_1\bar{S} + c_2\bar{S}^2 \\ &\quad + c_3\bar{\Omega}^2 \\ &\quad + c_4(\bar{S}\,\bar{\Omega} - \bar{\Omega}\,\bar{S}) + c_5(\bar{S}^2\bar{\Omega} \overset{\substack{\text{New term}\\ \downarrow}}{-} \bar{\Omega}\,\bar{S}^2)\end{aligned}$$

Improved representation of subgrid-scale stresses?

### What do the terms represent?

$$\begin{aligned}\mathcal{P}^{mod} = -\mathrm{tr}\left(\tau^{mod}\bar{S}\right) &= -c_1\mathrm{tr}(\bar{S}^2) \\ &\quad -c_2\mathrm{tr}(\bar{S}^3) \quad -c_4\mathrm{tr}(\bar{S}^2\bar{\Omega} - \bar{\Omega}\,\bar{S}^2) \\ &\quad -c_3\mathrm{tr}(\bar{S}\,\bar{\Omega}^2) \quad -c_5\mathrm{tr}(\bar{S}^3\bar{\Omega} - \bar{\Omega}\,\bar{S}^3)\end{aligned}$$

Energy transfer from large to small scales

Distribution of energy among large scales?

### How to set the coefficients?

- Require: $\tau_{ij} = 0 \Rightarrow \tau_{ij}^{mod} = 0$

  $\frac{\partial}{\partial x_j}\tau_{ij} = 0 \Rightarrow \frac{\partial}{\partial x_j}\tau_{ij}^{mod} = 0$

  Which flows satisfy this?

  Condition for some flows:

  $c_1 = 0,\ c_2 = c_3,\ c_4 = 0,\ c_5 = ?$

  How to determine $c_5$?
- Estimate from data:
  Projection pursuit regression?

### How to test the model?

## References

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