Constructing nonlinear subgrid-scale models for large-eddy simulation that preserve the symmetries of the Navier-Stokes equations

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Abstract In the current study we aim to go beyond the dissipative description of turbulent flows that is provided by eddy viscosity models for large-eddy simulation. We further aim to make this description consistent with a number of physical constraints. Starting from a general subgrid-scale model that is nonlinear in the velocity gradient, we show how a finite number of terms can be used to represent the subgrid-scale stresses. Then, several physical principles are outlined to determine the dependence of the model coefficients on flow properties. Ultimately, this would lead to a better understanding of the role of different nonlinear model terms in the description of turbulent flows.

Introduction

We study the construction of subgrid-scale models for large-eddy simulation of incompressible turbulent flows. In large-eddy simulation one seeks to predict the behavior of the larger scales of motion within a flow field. Usually, the distinction between large and small scales is made by a filtering or coarse-graining operation, and the evolution of the large-scale velocity field is given by the filtered Navier-Stokes equations,

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0, \qquad \frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}(\underbrace{\overline{u_i u_j} - \bar{u}_i \bar{u}_j}_{\tau_{ij}}). \tag{1}$$

The subgrid-scale stress tensor, τ, represents the interactions between large and small scales of motion. As it is not solely expressed in terms of the large-scale velocity field it cannot be resolved in a numerical simulation and it has to be modeled. The subgrid-scale models we consider here depend on the filtered rate-of-strain and rate-of-rotation tensors,

$$\bar{S}_{ij} = \frac{1}{2}(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}), \qquad \bar{\Omega}_{ij} = \frac{1}{2}(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}). \tag{2}$$

Examples of subgrid-scale models

Based on the idea that small-scale turbulent motions effectively cause diffusion of the larger scales, in eddy viscosity models the off-diagonal subgrid-scale stresses are often taken proportional to the rate of strain,

$$\tau^{mod} - \tfrac{1}{3}\,\mathrm{tr}(\tau^{mod})I = -2\nu_e \bar{S}. \tag{3}$$

When the eddy viscosity, ν_e, is chosen properly, this model captures the net transfer of energy from large to small scales. Unless ν_e is allowed to become negative, however, the reverse process of backscatter cannot be captured. Furthermore, the model incorrectly imposes alignment of the eigenvectors of the subgrid-scale stresses with those of the rate-of-strain tensor [1]. A different subgrid-scale model is the gradient model,

$$\tau^{mod} = C(\bar{S}^2 - \bar{\Omega}^2 - (\bar{S}\bar{\Omega} - \bar{\Omega}\bar{S})). \tag{4}$$

In several *a priori* studies it has been shown to capture the eigenvector orientations of subgrid-scale stresses better [1]. It is, however, unstable as it does not transport enough energy to smaller scales. Usually this problem is remedied by combining the above models, resulting in a mixed model by which forward and backward scatter can be represented. [2]

A general nonlinear subgrid-scale model

Motivated by the results provided by mixed models, we consider a general subgrid-scale model that is nonlinear in the velocity gradient. It is constructed by assuming that the subgrid-scale stress tensor can be expressed as an isotropic function of the filtered rate-of-strain and rate-of-rotation tensors, $\bar{S}$ and $\bar{\Omega}$. From the Cayley-Hamilton theorem it then follows that the most general model is a linear combination [3],

$$\tau^{mod} = \sum_{i=0}^{10} \alpha_i T_i, \tag{5}$$

of a finite number of tensors,

$$\begin{aligned}
&T_0 = I, && T_3 = \bar{\Omega}^2, && T_6 = \bar{S}\bar{\Omega}^2 + \bar{\Omega}^2\bar{S}, && T_9 = \bar{S}^2\bar{\Omega}^2 + \bar{\Omega}^2\bar{S}^2,\\
&T_1 = \bar{S}, && T_4 = \bar{S}\bar{\Omega} - \bar{\Omega}\bar{S}, && T_7 = \bar{\Omega}\bar{S}\bar{\Omega}^2 - \bar{\Omega}^2\bar{S}\bar{\Omega}, && T_{10} = \bar{\Omega}\bar{S}^2\bar{\Omega}^2 - \bar{\Omega}^2\bar{S}^2\bar{\Omega}.\\
&T_2 = \bar{S}^2, && T_5 = \bar{S}^2\bar{\Omega} - \bar{\Omega}\bar{S}^2, && T_8 = \bar{S}\bar{\Omega}\bar{S}^2 - \bar{S}^2\bar{\Omega}\bar{S}, &&
\end{aligned} \tag{6}$$

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The coefficients, α_i, may depend on the invariants of $\bar{S}$ and $\bar{\Omega}$,

$$\begin{aligned} I_1 &= \mathrm{tr}(\bar{S}^2), & I_3 &= \mathrm{tr}(\bar{S}^3), & I_5 &= \mathrm{tr}(\bar{S}^2\bar{\Omega}^2), \\ I_2 &= \mathrm{tr}(\bar{\Omega}^2), & I_4 &= \mathrm{tr}(\bar{S}\bar{\Omega}^2), & I_6 &= \mathrm{tr}(\bar{S}^2\bar{\Omega}^2\bar{S}\bar{\Omega}). \end{aligned} \tag{7}$$

The general model of Eqs. (5)–(7) is expected to allow for a better representation of subgrid-scale stresses than the eddy viscosity and gradient models, especially in wall-bounded and rotating flows. Containing as many as eleven adjustable constants, the model seems to be unnecessarily complicated, however. Indeed, as Lund and Novikov [3] remark, in most cases, the first six of the eleven tensors suffice to describe the degrees of freedom of the subgrid-scale stress tensor.

In practical tests usually a smaller set of terms is used. For instance, Marstorp et al. [4] derive a model consisting of three tensors from the evolution equation of the subgrid-scale stresses. Wang and Bergstrom [5] take a different set of four terms. For an extensive review of the use of these and similar nonlinear models in the RANS community, see [6].

In the present work, rather than discarding any of the above tensors on beforehand, we extend the analysis of [3] and [6], and perform a Gram-Schmidt orthogonalization process to isolate all independent contributions, say T_i'. In this fashion it is shown that, although usually the first six tensors suffice to form a representation of the subgrid-scale stress tensor, in some cases two others appear. The last three tensors never form an independent contribution to the model.

Physical model constraints

Having obtained a number of independent nonlinear model terms, T_i', we look to determine the functional dependence of their coefficients, α_i', on flow properties, based on analytical considerations.

First of all, the symmetries, or invariances, of the Navier-Stokes equations are considered. Requiring that they are preserved by the *filtered* Navier-Stokes equations leads to several conditions on the subgrid-scale model. [7, 8] We show how these conditions provide information on the dependence of the model coefficients on the invariants of Eq. (7).

Vreman [9] provides an extra requirement for the term linear in $\bar{S}$. He investigates the transport of energy from large to small scales and demands that for all flows for which it is zero, also the modeled energy transport vanishes,

$$-\tau_{ij}\bar{S}_{ij} = 0 \rightarrow -\tau_{ij}^{mod}\bar{S}_{ij} = 0. \tag{8}$$

We propose to extend Vreman's analysis to the case of the general nonlinear model by requiring that the modeled subgrid-scale force vanishes at all flow locations where there is no actual subgrid-scale force,

$$\frac{\partial}{\partial x_j}\tau_{ij} = 0 \rightarrow \frac{\partial}{\partial x_j}\tau_{ij}^{mod} = 0. \tag{9}$$

Other analytical criteria to restrict the model coefficients are under study, such as consistency with the second law of thermodynamics [8].

The above considerations lead to a class of nonlinear subgrid-scale models that possess certain exact properties of the actual subgrid-scale stresses. Numerical tests are planned to assess the behavior of specific models within this class in canonical turbulent flows. Ultimately, this would lead to a better understanding of the role of different nonlinear model terms in the description of turbulent flows.

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